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Energy dissipation in body-forced turbulence

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Bounds on the bulk rate of energy dissipation in body-force-driven steady-state turbulence are derived directly from the incompressible Navier–Stokes equations. We consider flows in three spatial dimensions in the absence of boundaries and derive rigorous *a priori* estimates for the time-averaged energy dissipation rate per unit mass, ϵ, without making any further assumptions on the flows or turbulent fluctuations. We prove

$$\epsilon \leqslant c_1 \nu \frac{U^2}{\ell^2} + c_2 \frac{U^3}{\ell},$$

where ν is the kinematic viscosity, U is the root-mean-square (space and time averaged) velocity, and ℓ is the longest length scale in the applied forcing function. The prefactors c_1 and c_2 depend only on the functional shape of the body force and not on its magnitude or any other length scales in the force, the domain or the flow. We also derive a new lower bound on ϵ in terms of the magnitude of the driving force F. For large Grashof number $Gr = F\ell^3/\nu^2$, we find

$$c_3 \frac{\nu F \ell}{\lambda^2} \leqslant \epsilon,$$

where $\lambda = \sqrt{\nu U^2/\epsilon}$ is the Taylor microscale in the flow and the coefficient c_3 depends only on the shape of the body force. This estimate is seen to be sharp for particular forcing functions producing steady flows with $\lambda/\ell \sim O(1)$ as $Gr \to \infty$. We interpret both the upper and lower bounds on ϵ in terms of the conventional scaling theory of turbulence – where they are seen to be saturated – and discuss them in the context of experiments and direct numerical simulations.

1. Introduction

The classical scaling theory of turbulence, as articulated in the first half of the twentieth century by Richardson, Taylor, Kolmogorov, etc., is based on the concept of the energy cascade. Energy is input on relatively large length scales determined by the initial data, the flow domain and boundary conditions, and/or the body forces driving the system. High Reynolds number turbulence obtains when the energy dissipation mechanism is ineffective on the length scales of the energy input, the so-called 'outer scale' of the flow. The nonlinear interaction – realized physically through the phenomenon of vortex stretching – transfers energy down to small length scales where viscosity dominates and the kinetic energy is dissipated into heat. For a modern review, see the book by Frisch (1995).

In this paper we focus on turbulence in a steadily driven unit-density incompressible Newtonian fluid in the absence of boundaries, in the long time limit. Then the initial data are ostensibly irrelevant and the energy flux balance is between the input at relatively large scale ℓ (a unit of *length*), defined by the forcing function, and a relatively small length scale where the viscosity ν (with units *length*^{2}/*time*) effectively dissipates the energy. In the simplest theoretical treatment, the flow is characterized by a single dominant velocity scale U (*length*/*time*) so the dimensionless Reynolds number is

$$Re = U\ell/\nu. \tag{1}$$

Laminar flows may be characterized by a rate of strain on the order of U/ℓ and stresses $\sim \nu U/\ell$, and thus an energy dissipation rate per unit mass (units *length*^{2}/*time*^{3})

$$\epsilon_{lam} \sim \nu(U/\ell)^2 = (U^3/\ell)Re^{-1}. \tag{2}$$

On the other hand, the rate of energy dissipation in turbulent flow corresponds to the rate of transfer of energy from large to small scales. Because the large-scale transfer mechanism is supposedly independent of the dissipation mechanism, the rate must be be proportional to the relatively large-scale eddy turn-over frequency $\sim U/\ell$. Hence turbulent energy dissipation should be proportional to this rate times the kinetic energy per unit mass, i.e.

$$\epsilon_{turb} \sim (U/\ell)U^2 = U^3/\ell. \tag{3}$$

Now consider an experiment in which the Reynolds number is ramped up from low to high values by, say, holding everything fixed except the viscosity which is decreased from large to small values. For the (thought) experiment in which the viscosity decreases without limit, the non-vanishing residual turbulent dissipation ϵ_{turb} indicates the presence of a broad 'inertial' range of length scales across which the kinetic energy flux flows. The expectation in this scenario is then that as Re varies from less than 1 towards ∞, the flow will cross over from a laminar state characterized by ϵ_{lam} to a turbulent one with overall dissipation ϵ_{turb} independent of the viscosity, i.e.

$$\epsilon \approx \frac{U^3}{\ell}\left(\frac{c_1}{Re} + c_2\right) \tag{4}$$

for some viscosity-independent constants c_1 and c_2.

Beyond the elementary dimensional analysis used in writing down (4), a great many unspoken assumptions have in fact been additionally implemented. For instance by restricting consideration solely to the length scale in the driving force, ℓ, we neglect the role of other large scales such as the domain size L (for a box of fluid of volume L^3 in three spatial dimensions), and other various familiar small scales (as we will define and use them in this paper):

the Taylor microscale $\lambda = (\nu U^2/\epsilon)^{1/2}$,

the Kolmogorov dissipation scale $\eta = (\nu^3/\epsilon)^{1/4}$,

or even other lengths like $\nu/U = Re^{-1}\ell$.

Moreover, the assumption of a single – yet to be quantitatively identified – velocity scale neglects the role of an emergent spectrum of velocities such as those that can be constructed (via dimensional analysis) by combining the magnitude of the forcing function, F (units *length*/*time*^{2}), with the various length scales in the domain, the forcing, or other Reynolds-number-dependent velocities generated in the flow itself. Finally, the form of the assumed scaling in (4) leaves open the possibility that the viscosity-independent constants c_1 and c_2 are not absolute constants; they may still depend on ratios of length scales (for example on the *aspect ratio* $\alpha = L/\ell$) and/or details of the functional form or *shape* of the body force. In the spirit of 'universality', currently fashionable in the physics literature concerning homogeneous turbulence, one might further expect that all the quantities in (4) are unambigously and uniquely defined independent of details of the driving force, and remain well behaved in the infinite volume limit $\alpha \to \infty$.

In this paper we examine the prediction of these heuristic considerations from an elementary albeit rigorous mathematical point of view, using only properties of the (weak) solutions of the Navier–Stokes equations of motion. We derive an upper bound on the bulk time-averaged energy dissipation rate of the form of (4), giving precise definitions to all the quantities involved. When U is the root-mean-square (space and time averaged) velocity and ℓ is the longest length scale in the applied body force (identified precisely in § 2), we find that the fundamental prediction is indeed valid as an upper limit on ϵ with c_1 and c_2 depending only on the shape of the body forcing function (also defined precisely below), independent of the overall system size L and all other parameters. Thus the estimate indeed retains its form in the limit $\alpha \to \infty$ and some of the expectations outlined above are in fact realized by the bound.

Rigorous asymptotic $Re \to \infty$ dissipation rate bounds of the form $\epsilon \leqslant CU^3/\ell$, where C is a viscosity and domain-volume-independent constant, were derived for a number of boundary-driven flows during the 1990s. The residual dissipation bound like this appeared for a shear layer – turbulent Taylor–Couette flow – where ℓ was the layer thickness and U was the overall velocity drop across the layer (Doering & Constantin 1992). That analysis, based on a mathematical device already introduced by Hopf (1941), is closely related to energy stability theory for stationary flows (Joseph 1976; Straughan 1992). Those results were extended to time-varying shear layers (Marchioro 1994) and more general domains (Wang 1997), and a variational method for optimizing the prefactor was introduced by Doering & Constantin (1994) and Constantin & Doering (1995*a*) and implemented (in part computationally) by Nicodemus, Grossmann & Holthaus (1998). A similar bound was recently derived for a shear layer with fluid injection and suction at the boundaries (Doering, Spiegel & Worthing 2000), a situation that is especially interesting because there is an exact solution that realizes the scaling of ϵ in the vanishing viscosity limit, establishing the sharpness of the result. This Hopf-inspired approach has also been applied to pressure-driven flows (Constantin & Doering 1995*b*), convection in the Boussinesq equations (Doering & Constantin 1996), and infinite Prandtl number (Constantin & Doering 1999) and porous medium convection (Doering & Constantin 1998). An earlier variational approach to the derivation of rigorous dissipation bounds given some mild statistical hypotheses, developed in the 1960s and 1970s by Howard (1972) and Busse (1978), has recently been shown to be intimately related (Kerswell 1998, 2001).

The first such rigorous limit on bulk dissipation for body-force-driven turbulence displaying a finite residual dissipation bound of the form U^3/ℓ in the vanishing viscosity limit was derived by Foias (1997). But that estimate did not have all the features of the form in (4); in particular the effective prefactor c_2 depended explicitly on $\alpha = L/\ell$ in that formulation. To a great extent the analysis here is to be considered a refinement of the approach by Foias (1997); we are now able to derive estimates which survive the infinite volume limit.

In a flow driven by a fixed body force the velocity scale U, and thus the Reynolds number, is not directly controllable. Rather, it is the amplitude of the force F and its dimensionless counterpart, the Grashof number

$$Gr = F\ell^3/\nu^2 \tag{5}$$

that may be specified *a priori*. While it has been observed that the rate of energy dissipation in turbulent flows tends towards the upper bounds *at fixed Reynolds number*, the opposite is the case at fixed (high) Grashof numbers. Then the estimate relevant for turbulent dissipation is a lower bound; consider, for example, a pressure-driven shear flow (Constantin & Doering 1995*b*) as compared to boundary-driven shear turbulence (Doering & Constantin 1994). In this paper we will show that as $Gr \to \infty$,

$$\epsilon \geqslant c_3 \frac{\nu F \ell}{\lambda^2}, \tag{6}$$

where c_3 depends only on the shape of the forcing function and λ is the Taylor microscale. We are not aware of as straightforward a physical argument as that described above for what the 'proper' scaling should be when $\nu \to 0$ and it is F rather than U that is held fixed. But it might be expected based on a cascade intuition – and indeed this is what is observed in numerical simulations (Borue & Orszag 1996; Childress, Kerswell & Gilbert 2001; Schumacher & Eckhardt 2000) – that the turbulent dissipation becomes independent of ν in the vanishing viscosity limit. The lower estimate in (6) is consistent with this if $\lambda/\ell = O(Gr^{-1/2})$ in the vanishing viscosity limit. This, in turn, requires that λ vanish at least as fast as $\nu^{1/2}$, which is both the conventional turbulence theory prediction *and* a rigorous lower bound on λ at fixed U (see further discussion about this in the conclusion of the paper). Hence an interesting conspiracy of saturated estimates is necessary for the lower bound in (6) to be considered as realized for turbulent flows.

To our knowledge the first such rigorous lower bounds of the form in (6) were presented by Foias, Manley & Temam (1993). But those results did not distinguish between L and ℓ or involve the Taylor microscale. Other lower estimates with this Grashof number scaling – again without the Taylor microscale dependence – are derived by Childress *et al.* (2001) in the context of a specific form (shape) of the body-forcing function. The results derived in this paper may be considered developments and extensions of those analyses.

The rest of this paper is organized as follows: Next, in § 2, we present definitions and the setting for the analysis. Then in § 3 we prove the central results of these considerations: Theorem 1, a rigorous version of the relationship in (4), and Theorem 2, the lower bound leading to (6). In the concluding § 4 we discuss these results in view of analytical results and both real and computational experiments. Elementary considerations produce, respectively, complementary lower and upper bounds, and exact (steady laminar) solutions establish the sharpness of the scalings in the estimates at large Gr and both small Gr and small Re. Direct numerical simulations provide a test of the quality of the estimates in the context of turbulent flows at high Re, and although there is no exact physical realization of the mathematical model that we analyse here, some of the results may be loosely interpreted in terms of homogeneous grid-generated turbulence in wind tunnels. In both cases we find that (even without direct correspondences between some quantities in the model and the theorem versus those in the turbulent simulations and experiments) the overall picture is consistent in general, and even in some particular details – notably the *non-universal* character of the constants in the scaling relations – with a saturation of the bounds by turbulent flows.

For the readers' convenience and in order to make the paper self-contained, we have included a brief appendix with a review of the elementary inequalities used to derive the results.

2. Set-up, notation and definitions

We consider the Navier–Stokes equations for the velocity vector field $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure field $p(\boldsymbol{x}, t)$ for an incompressible Newtonian fluid:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \nu \Delta \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x}), \quad \nabla \cdot \boldsymbol{u} = 0, \tag{7}$$

in a periodic box of side length L, i.e. with $\boldsymbol{x} \in \mathscr{T}^d$, the d-dimensional torus of volume L^d. (The analysis holds for $d = 2$ and 3, but we will focus on the applications in $d = 3$.) The applied body force $\boldsymbol{f}(\boldsymbol{x})$ is, without loss of generality, divergence free:

$$\nabla \cdot \boldsymbol{f} = 0. \tag{8}$$

We restrict attention to time-independent applied forces in this paper, but the analysis could be extended to a wide variety of time-dependent forces. The initial data for the velocity field are $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$, and without loss of generality we restrict attention to mean-zero body forces and initial conditions so the velocity remains mean-zero for all $t > 0$,

$$\int_{\mathscr{T}^d} \boldsymbol{u}(\boldsymbol{x}, t)\, \mathrm{d}^d x = 0. \tag{9}$$

We will frequently utilize the Fourier decomposition of the spatially dependent variables, using the conventions

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}(t) \tag{10}$$

for $\boldsymbol{k} = (2\pi/L)\boldsymbol{n}$ where $\boldsymbol{n} = (n_1, \ldots, n_d)$ with integer n_i and $k = |\boldsymbol{k}| \neq 0$. The Fourier coefficients are

$$\hat{\boldsymbol{u}}_{\boldsymbol{k}}(t) = \frac{1}{L^d} \int_{\mathscr{T}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}, t)\, \mathrm{d}^d x \tag{11}$$

so the divergence-free conditions in (7) and (8) are equivalent to the constraints

$$\boldsymbol{k} \cdot \hat{\boldsymbol{u}}_{\boldsymbol{k}}(t) = 0, \quad \boldsymbol{k} \cdot \hat{\boldsymbol{f}}_{\boldsymbol{k}} = 0. \tag{12}$$

The L^2 norms of derivatives of vector-valued functions will be denoted, for example, by

$$\|\nabla^N \boldsymbol{f}\|^2 = L^d \sum_{\boldsymbol{k}} k^{2N} |\hat{\boldsymbol{f}}_{\boldsymbol{k}}|^2. \tag{13}$$

This notation also holds for negative values of N, as defined by the term on the right-hand side above (remembering that $\boldsymbol{k} \neq 0$ in the sum).

For sufficiently regular body forces and initial data in $d = 3$, there are square-integrable weak solutions to these Navier–Stokes equations for all $t > 0$. This is Leray's 1933 existence result (Constantin & Foias 1988; Doering & Gibbon 1995). For clarity of presentation in this paper, we will manipulate the Navier–Stokes equations and solutions formally, keeping in mind that all of our results may indeed be justified in the strictest mathematical sense given appropriate attention to technical issues. For example, we will write the kinetic energy evolution by dotting $\boldsymbol{u}$ into the Navier–Stokes equation and integrating over $\mathscr{T}^d$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(\cdot, t)\|^2 = -\nu \|\nabla \boldsymbol{u}(\cdot, t)\|^2 + \int_{\mathscr{T}^d} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}, t)\, \mathrm{d}^d x, \tag{14}$$

although this is only known to hold as an inequality for the weak solutions.

Additionally, we will consider time-averaged quantities using the notation

$$\langle \mathscr{F}(\cdot) \rangle = \mathop{\mathrm{Lim}}_{t\to\infty} \frac{1}{t} \int_0^t \mathscr{F}(t')\,\mathrm{d}t' \tag{15}$$

where Lim indicates a generalized limit that exists for the weak 'statistical' solutions of the Navier–Stokes equations (Foias 1972, 1973; Foias & Prodi 1976); for modern developments see the book by Foias *et al.* (2001). As far as the formal calculations are concerned, the $\langle \cdot \rangle$ operation is the familar long-time averaging procedure.

Now Poincare's inequality, $\|\tilde{\boldsymbol{u}}(\cdot,t)\| \leqslant (L/2\pi)\|\nabla\tilde{\boldsymbol{u}}(\cdot,t)\|$, together with the Cauchy–Schwarz and Gronwall inequalities in (14) imply that the kinetic energy is uniformly bounded in time according to

$$\tfrac{1}{2}\|\boldsymbol{u}(\cdot,t)\|^2 \leqslant \tfrac{1}{2}\|\boldsymbol{u}_0\|^2 \exp(-4\pi^2\nu t/L^2) + \frac{L^2}{8\pi^2\nu^2}\|\nabla^{-1}\boldsymbol{f}\|^2(1-\exp(-4\pi^2\nu t/L^2)). \tag{16}$$

Hence the time average of the time derivative in (14) vanishes resulting in the formal power balance

$$\epsilon \equiv \frac{1}{L^d}\langle \nu\|\nabla\boldsymbol{u}\|^2\rangle = \left\langle \frac{1}{L^d}\int_{\mathscr{T}^d} \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x},\cdot)\,\mathrm{d}^d x \right\rangle \tag{17}$$

where we have introduced the definition of the long-time-averaged energy dissipation rate per until mass, ϵ. The root-mean-square (r.m.s.) velocity is defined by

$$U = \sqrt{\left\langle \frac{1}{L^d}\|\boldsymbol{u}\|^2 \right\rangle}. \tag{18}$$

Finally, we consider the detailed structure of the body-force functions we will use in the analysis here. Let $\mathscr{I}^d$ denote the unit d-torus ($[0,1]^d$ with periodic boundary conditions) and $\Phi(\boldsymbol{y})$ be a normalized, mean-zero, divergence-free vector field on $\mathscr{I}^d$;

$$1 = \int_{\mathscr{I}^d} |\nabla_y^{-1}\Phi(\boldsymbol{y})|^2\,\mathrm{d}^d y, \tag{19}$$

$$0 = \int_{\mathscr{I}^d} \Phi(\boldsymbol{y})\,\mathrm{d}^d y, \tag{20}$$

$$0 = \frac{\partial\Phi_1}{\partial y_1} + \cdots + \frac{\partial\Phi_d}{\partial y_d}. \tag{21}$$

Now let $\ell = L/\alpha$ for some integer α (the *aspect ratio*) and consider the mean-zero divergence-free body-force functions $\boldsymbol{f}(\boldsymbol{x})$ on $\mathscr{T}^d$ defined by

$$\boldsymbol{f}(\boldsymbol{x}) = F\Phi(\ell^{-1}\boldsymbol{x}). \tag{22}$$

See figure 1 for a graphic illustration for the set-up. We will refer to F as the *amplitude* of the applied force, to ℓ as the (longest) *length scale* in the force, and to $\Phi(\boldsymbol{y})$ for $\boldsymbol{y} \in \mathscr{I}^d$ as the *shape* of the force.

The L^2 norms of derivatives of $\boldsymbol{f}$ on $\mathscr{T}^d$ are

$$\|\nabla^N\boldsymbol{f}\|^2 = C_N\ell^{-2N}F^2L^d, \tag{23}$$

where the coefficients C_N depend only on the shape of the force according to

$$C_N \equiv \sum_{\boldsymbol{n}} |2\pi\boldsymbol{n}|^{2N}|\hat{\Phi}_{\boldsymbol{n}}|^2. \tag{24}$$

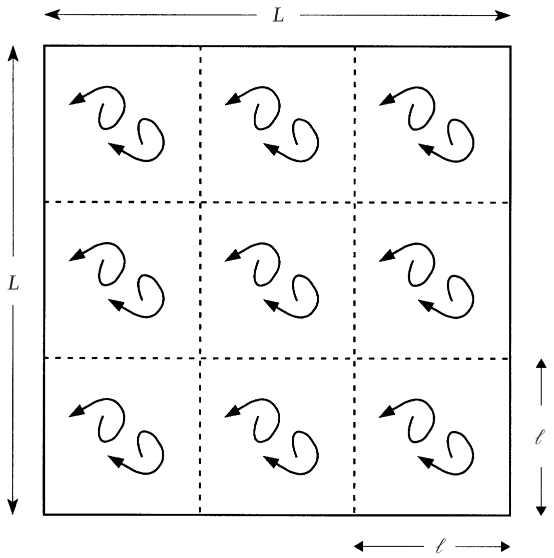

FIGURE 1. The flow domain is the periodic box of size L^d (with $d = 3$ or $d = 2$ as illustrated here) while the applied body force is periodic on an independent scale ℓ. The aspect ratio is an integer $\alpha = L/\ell \geqslant 1$.

Here $\boldsymbol{n} = (n_1, \ldots, n_d)$ with integers n_i, $|\boldsymbol{n}| \neq 0$ and $\hat{\Phi}_{\boldsymbol{n}}$ are the Fourier coefficients

$$\hat{\Phi}_{\boldsymbol{n}} = \int_{\mathscr{I}^d} e^{-i2\pi \boldsymbol{n}\cdot\boldsymbol{y}} \Phi(\boldsymbol{y})\, d^d y. \tag{25}$$

Note that $C_{-1} = 1$ and the definition (23) makes sense for N sufficiently small or sufficiently negative, if not for all N.

We will also utilize some sup-norms of the shape function. For values of M where the right-hand side is finite, we define

$$D_M \equiv \sup_{\boldsymbol{y}\in\mathscr{I}^d} |\nabla_y \Delta_y^{-M} \Phi(\boldsymbol{y})|, \quad E_M \equiv \sup_{\boldsymbol{y}\in\mathscr{I}^d} |\Delta_y^{-M} \Phi(\boldsymbol{y})| \tag{26}$$

so that

$$\|\nabla \Delta^{-M} \boldsymbol{f}\|_\infty = D_M F \ell^{2M-1}, \quad \|\Delta^{-M} \boldsymbol{f}\|_\infty = E_M F \ell^{2M}. \tag{27}$$

The coefficients D_M and E_M, like the constants C_N, depend only on the shape of the force function. Moreover, no matter what the spatial dimension d is or how slowly the coefficients $|\hat{\Phi}_{\boldsymbol{n}}|$ decay as $|\boldsymbol{n}| \to \infty$, there is always an $\mathscr{M} < \infty$ for which $M > \mathscr{M}$ implies D_M and E_M are finite. For instance if it happens to be the case that $\Phi \in L^2(\mathscr{I}^d)$, then $\mathscr{M} = (d+2)/4$ certainly works.

3. Energy dissipation rate estimates

In this section we will derive the basic relationships used to establish the fundamental results, and then prove the key theorems providing the estimates of ϵ in terms of ν, U, F and ℓ.

Recall first the defining balance for ϵ in (17), using (22) to write

$$\epsilon = F \left\langle \frac{1}{L^d} \int_{\mathscr{T}^d} \Phi(\ell^{-1}\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}, \cdot)\, d^d x \right\rangle. \tag{28}$$

Then, for integer M (M is arbitrary for Galerkin approximations but take $M > \frac{1}{2}$ for the weak solutions), multiply the Navier–Stokes equations by $(-\Delta)^{-M}\boldsymbol{f}$, integrate over $\mathscr{T}^d$ and integrate by parts as appropriate to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{T}^d} \boldsymbol{u}\cdot((-\Delta)^{-M}\boldsymbol{f})\,\mathrm{d}^d x = -\nu\int_{\mathscr{T}^d} \boldsymbol{u}\cdot((-\Delta)^{-M+1}\boldsymbol{f})\,\mathrm{d}^d x + \int_{\mathscr{T}^d} \boldsymbol{u}\cdot(\nabla(-\Delta)^{-M}\boldsymbol{f})\cdot\boldsymbol{u}\,\mathrm{d}^d x + \|\nabla^{-M}\boldsymbol{f}\|^2. \quad (29)$$

Now take the time average, noting that the time average of the time derivative vanishes, and divide by L^d to deduce

$$C_{-M}\ell^{2M}F^2 = F\left\langle \frac{1}{L^d}\int_{\mathscr{T}^d}\left[\nu\boldsymbol{u}\cdot((-\Delta)^{-M+1}\Phi) - \boldsymbol{u}\cdot(\nabla(-\Delta)^{-M}\Phi)\cdot\boldsymbol{u}\right]\mathrm{d}^d x\right\rangle. \quad (30)$$

In this equation Φ means $\Phi(\ell^{-1}\boldsymbol{x})$. Combining (28) and (30), we have established

LEMMA 1. *The energy dissipation rate per unit mass is given in terms of averages of the velocity field, the length scale in the body force, and the shape function by*

$$\epsilon = \frac{1}{\ell^{2M}C_{-M}}\left\langle \frac{1}{L^d}\int_{\mathscr{T}^d}\Phi\cdot\boldsymbol{u}\,\mathrm{d}^d x\right\rangle \times\left\langle \frac{1}{L^d}\int_{\mathscr{T}^d}\left[\nu\boldsymbol{u}\cdot((-\Delta)^{-M+1}\Phi) + \boldsymbol{u}\cdot(-\nabla(-\Delta)^{-M}\Phi)\cdot\boldsymbol{u}\right]\mathrm{d}^d x\right\rangle. \quad (31)$$

The relation (31) already posseses the proper homogeneity with respect to the system size L and the length scale in the force, ℓ. What remains is to make the scalings explicit.

First, note that the Cauchy–Schwarz inequality implies that

$$\left\langle \frac{1}{L^d}\int_{\mathscr{T}^d}\Phi\cdot\boldsymbol{u}\,\mathrm{d}^d x\right\rangle \leqslant \sqrt{\frac{1}{L^d}\|\Phi\|^2}\times\sqrt{\frac{1}{L^d}\langle\|\boldsymbol{u}\|^2\rangle} = \sqrt{C_0}U. \quad (32)$$

In the final step above we used the definition (18) and the fact that, according to (24),

$$\frac{1}{L^d}\|\Phi\|^2 = C_0. \quad (33)$$

Next, use Cauchy–Schwarz inequality together with (23) to see that

$$\left\langle \frac{1}{L^d}\int_{\mathscr{T}^d}\nu\boldsymbol{u}\cdot((-\Delta)^{-M+1}\Phi)\,\mathrm{d}^d x\right\rangle \leqslant \nu U\frac{1}{L^{d/2}}\|\nabla^{2-2M}\Phi\| = \nu U\ell^{2M-2}\sqrt{C_{2-2M}}. \quad (34)$$

Finally, recalling (26), choosing M sufficiently large, and utilizing Hölder's inequality we have

$$\left\langle \frac{1}{L^d}\int_{\mathscr{T}^d}\boldsymbol{u}\cdot(-\nabla(-\Delta)^{-M}\Phi)\cdot\boldsymbol{u}\,\mathrm{d}^d x\right\rangle \leqslant D_M\ell^{2M-1}U^2. \quad (35)$$

Combining the identity from Lemma 1 with the estimates in (32), (34) and (35),

$$\epsilon \leqslant \frac{\sqrt{C_0}U}{\ell^{2M}C_{-M}}\times(\nu U\ell^{2M-2}\sqrt{C_{2-2M}} + D_M U^2\ell^{2M-1}) = \frac{\sqrt{C_0 C_{2-2M}}}{C_{-M}}\nu\frac{U^2}{\ell^2} + \frac{\sqrt{C_0}D_M}{C_{-M}}\frac{U^3}{\ell}. \quad (36)$$

We have thus proven one of the central results of this paper which we summarize as

THEOREM 1. *Suppose* $\boldsymbol{y} \in \mathscr{I}^d = [0,1]^d$ *with periodic boundary conditions and* $\Phi(\boldsymbol{y}) \in L^2(\mathscr{I}^d)$ *is a divergence-free vector field with mean zero and* $\nabla^{-1}\Phi(\boldsymbol{y}) \equiv \nabla\Delta^{-1}\Phi(\boldsymbol{y})$ *has norm 1 in* $L^2(\mathscr{I}^d)$. *Let* $L = \alpha\ell$ *for some integer* α *and* $\boldsymbol{x} \in \mathscr{T}^d = [0,L]^d$ *with periodic boundary conditions, and* $\boldsymbol{u}(\boldsymbol{x},t)$ *be a mean-zero solution of the Navier–Stokes equations*

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x}), \quad \nabla\cdot\boldsymbol{u} = 0, \tag{37}$$

with body force $\boldsymbol{f}(\boldsymbol{x})$ *given by*

$$\boldsymbol{f}(\boldsymbol{x}) = F\Phi(\ell^{-1}\boldsymbol{x}). \tag{38}$$

Then the time-averaged energy dissipation rate per unit mass,

$$\epsilon \equiv \operatorname*{Lim}_{t\to\infty} \frac{1}{t}\int_0^t \nu \frac{1}{L^d}\|\nabla\boldsymbol{u}(\cdot,t')\|^2_{L^2(\mathscr{T}^d)}\,\mathrm{d}t', \tag{39}$$

satisfies

$$\epsilon \leqslant\ a_M \nu\frac{U^2}{\ell^2} + b_M\frac{U^3}{\ell}, \tag{40}$$

where the space–time-averaged root-mean-square velocity U *is defined by*

$$U^2 \equiv \operatorname*{Lim}_{t\to\infty} \frac{1}{t}\int_0^t \frac{1}{L^d}\|\boldsymbol{u}(\cdot,t')\|^2_{L^2(\mathscr{T}^d)}\,\mathrm{d}t', \tag{41}$$

and the coefficients a_M *and* b_M *– uniform in the parameters* ν, F, ℓ, L *and* α *– are*

$$a_M = \frac{\|\Phi\|_{L^2(\mathscr{I}^d)}\|\Delta_y^{1-M}\Phi\|_{L^2(\mathscr{I}^d)}}{\|\nabla_y^{-M}\Phi\|^2_{L^2(\mathscr{I}^d)}}, \tag{42}$$

$$b_M = \frac{\|\Phi\|_{L^2(\mathscr{I}^d)}\sup_{\boldsymbol{y}\in\mathscr{I}^d}|\nabla_y\Delta_y^{-M}\Phi(\boldsymbol{y})|}{\|\nabla_y^{-M}\Phi\|^2_{L^2(\mathscr{I}^d)}}. \tag{43}$$

Now we turn to the derivation of lower bounds on ϵ in terms of the viscosity and the parameters of the applied force. There are two nearly parallel approaches we will take, one of which yields a precise estimate for low values of Gr, and the other which is potentially more effective in the limit of turbulence at high Gr (these remarks will be elaborated in the discussion in the next section).

Return to (30), integrate by parts once each of the terms on the right-hand side, and estimate them with the help of the Cauchy–Schwarz and Hölder inequalities:

$$\begin{aligned} C_{-M}\ell^{2M}F^2 &= F\left\langle \frac{1}{L^d}\int_{\mathscr{T}^d}\left[\nu\nabla\boldsymbol{u}:\left(\nabla\Delta^{-M}\Phi\right) + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\left(\Delta^{-M}\Phi\right)\right]\mathrm{d}^d x\right\rangle \\ &\leqslant \sqrt{C_{1-2M}}\ F\ell^{2M-1}\nu^{1/2}\epsilon^{1/2} + E_M F\ell^{2M}U\left(\frac{\epsilon}{\nu}\right)^{1/2}. \end{aligned} \tag{44}$$

Then replace the U on the right-hand side with $\sqrt{\epsilon\lambda^2/\nu}$, directly from the definition of the Taylor microscale λ. A little algebra, utilizing the definition of the Grashof number $Gr = F\ell^3/\nu^2$, subsequently yields

$$C_{-M}Gr^{-1/2} \leqslant \sqrt{C_{1-2M}}Gr^{-3/4}\left(\frac{\epsilon}{F^{3/2}\ell^{1/2}}\right)^{1/2} +\ E_M\ \frac{\lambda}{\ell}\frac{\epsilon}{F^{3/2}\ell^{1/2}}. \tag{45}$$

Solving for $\epsilon/(F^{3/2}\ell^{1/2})$, we see that

$$\frac{\epsilon}{F^{3/2}\ell^{1/2}} \geqslant \frac{\ell}{\lambda}Gr^{-1/2}\left(\frac{C_{-M}}{E_M} + \frac{C_{1-2M}}{2E_M^2}\frac{\ell}{\lambda}Gr^{-1}\left[1-\sqrt{1+\frac{4C_{-M}E_M}{C_{1-2M}}\frac{\lambda}{\ell}Gr}\right]\right). \tag{46}$$

This may be transformed into a more explicit bound without any reference to the Taylor microscale by noting that Poincare's inquality implies $\lambda \leqslant L/2\pi$ so $\ell/\lambda \geqslant 2\pi/\alpha$. Note also that as $Gr \to 0$, this estimate simplifies to

$$\epsilon \geqslant \frac{C_{-M}^2}{C_{1-2M}}\frac{F^2\ell^2}{\nu} \tag{47}$$

independent of the Taylor microscale λ and/or the aspect ratio α.

A slightly different bound is obtained by returning again to (30) and invoking the estimates in (34) and (35) to deduce an explicit lower bound on U:

$$C_{-M}F \leqslant \sqrt{C_{2-2M}}\ell^{-2}\nu U + D_M\ell^{-1}U^2. \tag{48}$$

Dividing through by F, rearranging and solving for $U/\sqrt{F\ell}$, we find

$$\frac{U}{\sqrt{F\ell}} \geqslant \sqrt{\frac{C_{2-2M}}{4D_M^2 Gr} + \frac{C_{-M}}{D_M}} - \sqrt{\frac{C_{2-2M}}{4D_M^2 Gr}}. \tag{49}$$

This may be re-expressed as a lower limit on ϵ simply by squaring and substituting $\epsilon\lambda^2/\nu$ for U^2:

$$\frac{\epsilon\lambda^2}{\nu F\ell} \geqslant \frac{C_{-M}}{D_M} + \frac{C_{2-2M}}{2D_M^2}Gr^{-1}\left[1-\sqrt{1+\frac{4C_{-M}D_M}{C_{2-2M}}Gr}\right]. \tag{50}$$

From this result it is then straightforward to see that as $Gr \to \infty$,

$$\epsilon \geqslant \frac{C_{-M}}{D_M}\frac{\nu F\ell}{\lambda^2} \tag{51}$$

which is (6) as expected.

We collect these results into

THEOREM 2. *Suppose $\boldsymbol{y} \in \mathscr{I}^d = [0,1]^d$ with periodic boundary conditions and $\Phi(\boldsymbol{y})$ is a divergence-free vector field with mean-zero and $\nabla^{-1}\Phi(\boldsymbol{y})$ has norm 1 in $L^2(\mathscr{I}^d)$. Let $L = \alpha\ell$ for some integer α and $\boldsymbol{x} \in \mathscr{T}^d = [0,L]^d$ with periodic boundary conditions, and $\boldsymbol{u}(\boldsymbol{x},t)$ be a mean-zero solution of the Navier–Stokes equations*

$$\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x}), \quad \nabla\cdot\boldsymbol{u} = 0, \tag{52}$$

with body force $\boldsymbol{f}(\boldsymbol{x})$ given by

$$\boldsymbol{f}(\boldsymbol{x}) = F\Phi(\ell^{-1}\boldsymbol{x}). \tag{53}$$

Then the long-time-averaged energy dissipation rate per unit mass, ϵ satisfies both

$$\epsilon \geqslant c_M\frac{\nu F}{\ell}\frac{\ell}{\lambda}\left(1+\frac{d_M}{2Gr}\frac{\ell}{\lambda}\left[1-\sqrt{1+\frac{4Gr}{d_M}\frac{\lambda}{\ell}}\right]\right) \tag{54}$$

and

$$\epsilon \geqslant e_M\frac{\nu F}{\ell}\frac{\ell^2}{\lambda^2}\left(1+\frac{f_M}{2Gr}\left[1-\sqrt{1+\frac{4Gr}{f_M}}\right]\right), \tag{55}$$

where Gr is the Grashof number

$$Gr \equiv \frac{F\ell^3}{\nu^2}, \tag{56}$$

λ is the Taylor microscale defined by

$$\lambda^2 = \frac{\nu U^2}{\epsilon}, \tag{57}$$

and the coefficients c_M, d_M, e_M and f_M – uniform in the parameters ν, F, ℓ, L and α – are

$$c_M = \frac{\|\nabla_y^{-M}\Phi\|^2_{L^2(\mathscr{I}^d)}}{\sup_{y\in\mathscr{I}^d}|\Delta_y^{-M}\Phi(\boldsymbol{y})|}, \tag{58}$$

$$d_M = \frac{\|\nabla_y\Delta_y^{-M}\Phi\|^2_{L^2(\mathscr{I}^d)}}{\sup_{y\in\mathscr{I}^d}|\Delta_y^{-M}\Phi(\boldsymbol{y})| \times \|\nabla_y^{-M}\Phi\|^2_{L^2(\mathscr{I}^d)}}, \tag{59}$$

$$e_M = \frac{\|\nabla_y^{-M}\Phi\|^2_{L^2(\mathscr{I}^d)}}{\sup_{y\in\mathscr{I}^d}|\nabla\Delta_y^{-M}\Phi(\boldsymbol{y})|}, \tag{60}$$

$$f_M = \frac{\|\Delta_y^{-M+1}\Phi\|^2_{L^2(\mathscr{I}^d)}}{\sup_{y\in\mathscr{I}^d}|\nabla\Delta_y^{-M}\Phi(\boldsymbol{y})| \times \|\nabla_y^{-M}\Phi\|^2_{L^2(\mathscr{I}^d)}}. \tag{61}$$

For general shape functions with merely square-integrable $\nabla^{-1}\Phi(\boldsymbol{y})$, $M > \frac{1}{2}$ at least. For smoother shape functions M may be smaller, even negative.

4. Discussion

Several remarks are in order concerning the estimates for ϵ in Theorem 1, which we rewrite here in terms of the Reynolds number $Re = U\ell/\nu$ and the dimensionless *dissipation ratio*

$$\beta \equiv \frac{\epsilon\ell}{U^3} \leqslant \left(\frac{a_M}{Re} + b_M\right). \tag{62}$$

Figure 2 is an illustration of the upper bound in this form. Theorem 1 proves that the dissipation ratio remains uniformly bounded in both the infinite Reynolds number and the infinite volume limits as anticipated by heuristic scaling arguments outlined in the introduction. The bound on the asymptotic $Re \to \infty$ value of the dissipation ratio, $b_M = O(1)$, depends only on the shape $\Phi(\boldsymbol{y})$ of the driving force. For a given body-force shape function, the parameter M may in principle be adjusted in a Re-dependent manner to minimize the right-hand side of (62). This small amount of freedom in the choice of M appears to lead at most to a quantitative improvement of the bound. Hence to simplify notation in the remainder of this section, we will for the most part drop the indices on shape-dependent constants (a, b, c, C, C', etc.).

Often in applications the Taylor–Reynolds number $R_\lambda = U\lambda/\nu$ based on the Taylor microscale is used to indicate the intensity of the turbulence. The wavenumber associated with this scale, λ^{-1}, is the standard deviation of the Fourier power spectrum of the flow field. Hence when there is a broad range of length scales in a turbulent flow, λ is an intermediate scale in the so-called *inertial* range. We may restate the

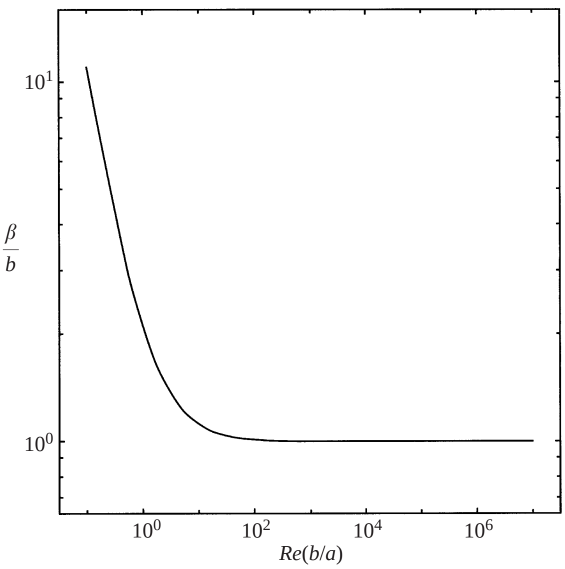

FIGURE 2. Illustration of the bound on the dissipation factor $\beta = \epsilon\ell/U^3$ as a function of the Reynolds number $Re = U\ell/\nu$. The scale factors a and b depend only on the shape of the applied body force. This kind of functional relationship, as well as a dependence of scale factors on the details of the forcing, is realized by turbulent flows (see in particular Sreenivasan's (1998) summary of data from direct numerical simulations).

result of Theorem 1 in terms of R_λ by bounding the Taylor microscale from below:

$$\lambda = \left(\frac{\nu U^2}{\epsilon}\right)^{1/2} \geqslant \ell\left(Re\left(\frac{a}{Re}+b\right)\right)^{-1/2}. \tag{63}$$

Hence

$$R_\lambda \geqslant \sqrt{\frac{Re}{((a/Re)+b)}} \tag{64}$$

and the Taylor–Reynolds number grows at least as fast as $Re^{1/2}$ as $Re \to \infty$, whether the flow is turbulent or not. The classical theory of turbulence predicts that precisely this bound's scaling, $R_\lambda \sim Re^{1/2}$, is realized at high Reynolds number.

The upper bound on dissipation ratio β estimate in (62) is actually observed at high Reynolds number in a wide variety of experimental and computational studies. This is generally understood in terms of the cascade picture of turbulent dynamics mentioned in the introduction: the energy dissipation rate tends to become independent of the viscosity in high-Reynolds-number turbulence. Conveniently, Sreenivasan has collected a number of measurements of β from wind tunnels (Sreenivasan 1984) and direct numerical simulations (Sreenivasan 1998), providing plots of the dissipation factor β analogous to that in figure 2. The data indicate that for R_λ in excess of between 50 and 100, corresponding to Re in the thousands and beyond, β is a constant of order 1. Measured asymptotic values of β range from system to system between about 0.5 and 3 depending on the details of the form of the body forcing (in simulations) or the grid structure (in the wind tunnels), corresponding nicely with the body-force shape dependence of the constants anticipated by the analysis in this paper.

The bound on the dissipation ratio in (62) may be recast as an upper bound on β in terms of R_λ, as the data are often plotted (Sreenivasan 1984, 1998). To see this, note that by the definitions of β, Re, λ and R_λ we have the simple identity $Re = \beta R_\lambda^2$. Then substitute βR_λ^2 for Re in (62) and solve for β to see that

$$\beta \leqslant \frac{b}{2}\left(1+\sqrt{1+\frac{4a}{b^2}\frac{1}{R_\lambda^2}}\right). \tag{65}$$

For decreasing values of R_λ, necessarily corresponding to decreasing values of Re, β increases in all experiments and simulations, consistent with the crossover from turbulent dissipation ($\beta \sim Re^0$ or $\sim R_\lambda^0$) to laminar scaling ($\beta \sim Re^{-1}$ or $\sim R_\lambda^{-1}$). Indeed, the low-Reynolds-number scalings in (62) and (65) are sharp as seen by an application of Poincare's inequality ($\|\nabla \boldsymbol{u}\|^2 \geqslant (4\pi^2/L^2)\|\boldsymbol{u}\|^2 \Longrightarrow \beta \geqslant (4\pi^2/\alpha^2 Re)$). Hence we have the limits, uniform in Re for all values of the Reynolds number,

$$\frac{4\pi^2}{\alpha^2 Re} \leqslant \beta \leqslant \left(\frac{a}{Re}+b\right). \tag{66}$$

In terms of R_λ, we have thus proven

$$\frac{2\pi}{\alpha R_\lambda} \leqslant \beta \leqslant \frac{b}{2}\left(1+\sqrt{1+\frac{4a}{b^2}\frac{1}{R_\lambda^2}}\right). \tag{67}$$

The only significant difference between the upper and lower bounds in both (66) and (67) as the Reynolds numbers $\to 0$ is in the aspect ratio dependence of the prefactors.

Now we turn to point out precisely what the bounds on ϵ in (66) and (67) do *not* tell us: for a given applied body force they yield no *a priori* information on the magnitude or structure of the resulting flow field or dissipation rate. Consider again the experimental scenario described in the introduction where all the system variables are held fixed except the viscosity, which is lowered without limit in search of high-Reynolds-number turbulence. As we have formulated the problem, the parameters under our control are L, ν, F, ℓ and the body-force shape function $\Phi(\boldsymbol{y})$. The estimates for ϵ in (66) are expressed in terms of ℓ and ν, controlled quantities, and the r.m.s. velocity U, a derived quantity. In (67) they are expressed in terms of λ, another derived quantity. But we do not know how to realize a specific value of U (or ϵ) or how to guarantee a particular Re or R_λ for these flows; as we have designed the experiment the Grashof number is the proper control parameter, not the Reynolds number.

The results of Theorem 2 are just the type of lower estimate on ϵ that provide a partial answer to the question of how large a 'response' is to be expected from a given applied force. The appropriate dimensionless expression for ϵ that is independent of ν at *a priori* specified F (the analogue of the dissipation factor β at a given value of U) is

$$\gamma \equiv \epsilon/\sqrt{F^3\ell}, \tag{68}$$

and the bounds in Theorem 2 may be rephrased as a lower bounds on γ:

$$\gamma \geqslant c Gr^{-1/2}\frac{\ell}{\lambda}\left(1+\frac{d}{2Gr}\frac{\ell}{\lambda}\left[1-\sqrt{1+\frac{4Gr}{d}\frac{\lambda}{\ell}}\right]\right) \tag{69}$$

and

$$\gamma \geqslant c' Gr^{-1/2} \frac{\ell^2}{\lambda^2} \left(1 + \frac{d'}{2Gr} \left[1 - \sqrt{1 + \frac{4Gr}{d'}} \right] \right). \tag{70}$$

Let us begin by examining these estimates in the limit $Gr \to 0$. In order to clarify their contents, note that the function $\psi(x) = (1/x)(1 + (1/2x)[1 - \sqrt{1+4x}])$ decreases monotonically from 1 to $O(1/x)$ as x varies from 0 to $+\infty$. Then use $\lambda \leqslant L/2\pi \Rightarrow \ell/\lambda \geqslant 2\pi/\alpha$ to see that the $Gr \to 0$ limits of these are, respectively,

$$\gamma \geqslant \frac{c}{d} Gr^{1/2} = \frac{C_{-M}^2}{C_{1-2M}} Gr^{1/2} \tag{71}$$

and

$$\gamma \geqslant \frac{c'}{d'} Gr^{1/2} \frac{\ell^2}{\lambda^2}. \tag{72}$$

Not unexpectedly, we can also estimate γ from above in terms of Gr. Indeed, for any divergence-free vector field constrained by the balance in (17),

$$\left\langle \nu \|\nabla \boldsymbol{u}\|^2 \right\rangle = \left\langle \int_{\mathscr{T}^d} \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}^d x \right\rangle, \tag{73}$$

the dissipation rate obeys

$$\left\langle \nu \|\nabla \boldsymbol{u}\|^2 \right\rangle \leqslant \nu \|\nabla \boldsymbol{U}_{St}\|^2 \equiv L^d \epsilon_{Stokes}, \tag{74}$$

where $\boldsymbol{U}_{St}(\boldsymbol{x})$ is the solution of the Stokes equation

$$-\nu \Delta \boldsymbol{U}_{St} + \nabla P = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{U}_{St} = 0. \tag{75}$$

Thus

$$\epsilon \leqslant \epsilon_{Stokes} = \frac{F^2 \ell^2}{\nu} \sum_{\boldsymbol{n} \neq 0} \frac{|\hat{\Phi}_{\boldsymbol{n}}|^2}{4\pi^2 n^2} = C_{-1} \frac{F^2 \ell^2}{\nu} \Rightarrow \gamma \leqslant C_{-1} Gr^{1/2}. \tag{76}$$

Therefore, recalling that $C_{-1} \equiv 1$ and using $M = 1$ in (71), we see that the upper bound in (76) and lower bound in (69) actually converge at low Gr:

$$\lim_{Gr \to 0} \gamma = Gr^{1/2} \Longleftrightarrow \lim_{Gr \to 0} \epsilon = \frac{F^2 \ell^2}{\nu}. \tag{77}$$

Combined with the upper bound in (76), the alternative lower bound in (72) implies simply that as $Gr \to 0$,

$$\lambda \geqslant C\ell \Longleftrightarrow \beta \leqslant C \frac{1}{Re} \tag{78}$$

(where $C = \sqrt{e_M/f_M}$).

When the Grashof number is fixed, the energy dissipation rate is maximized by a Stokes flow. This is distinct from the dissipation rate's minimization by such a laminar flow at fixed Reynolds number; the low-*Re* limiting Stokes flow realizes the lower bound in (66), modulo the aspect ratio factor. For some simple shapes of the applied force the extremizing Stokes flow may be an exact – albeit possibly unstable – solution of the Navier–Stokes equations at arbitrary Grashof [or Reynolds] number, so the upper bound in (76) [or lower bound in (66)] may actually be sharp at all values of *Gr* [*Re*].

The lower bounds on γ in (69) and (70) are also saturated by steady laminar flows at high values of *Gr* (Foias *et al.* 1993). To see this, choose any divergence-free velocity field $\boldsymbol{v}(\boldsymbol{x})$ periodic on the length scale ℓ that is not a solution of the homogeneous Euler equations (i.e. such that $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ is not a gradient). Then define divergence-free $\boldsymbol{f}(\boldsymbol{x})$ by $\boldsymbol{f}=\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\nabla p-\nu\Delta\boldsymbol{v}$ so that the resulting r.m.s. velocity scale U and the force magnitude F are related according to $F\sim U^2/\ell$ as $\nu\to 0$, corresponding precisely to (48) as $Gr\to\infty$. For such flows $\lambda=O(\ell)$ uniformly in Gr, so $\gamma\sim Gr^{-1/2}$ consistent with the lower limits in (69) and (70) at high Gr. Note that this latter example does not saturate the upper bound on ϵ in Theorem 1 or (66) or (67); that particular limit seems to require truly turbulent flows to be realized.

On the other hand, direct numerical simulations of turbulent body-forced flows (Borue & Orszag 1996; Childress *et al.* 2001; Schumacher & Eckhardt 2000) indicate that $\gamma\sim Gr^0$ as $Gr\to\infty$. This is the expectation in accord with fundamental experimental 'law' of turbulence that the energy dissipation rate becomes independent of viscosity as $\nu\to 0$, all other parameters held fixed (Frisch 1995). This observed $\gamma\sim Gr^0$ might also be conjectured to be the scaling in the 'best' lower bound, for while turbulence tends to saturate the upper bound on ϵ at fixed Re it should correspondingly lean toward the lower limit at fixed Gr.

Indeed, the Grashof number may be considered a measure of the applied force (non-dimensionalized by ν and ℓ) and the Reynolds number a measure of the velocity in the flow (also non-dimensionalized by ν and ℓ). Disregarding details of the spatial correlations between the shape of the force function and the structure of the mean flow field, the product $Gr\times Re$ is a measure of the dissipated power ϵ (non-dimensionalized by ν and ℓ). Laminar flows subject only to molecular friction forces are expected to maximize Re at fixed Gr; equivalently they should minimize Gr at fixed Re. The enhanced dissipation associated with turbulence means that turbulent flows should reduce (minimize?) Re at fixed Gr, relative to laminar flows at the same value of Gr. Equivalently turbulence should increase (maximize?) Gr at fixed Re. These considerations imply just what was asserted: turbulence may be expected to tend toward the minimum of $\epsilon\times(\ell^4/\nu^3)\sim Gr\times Re$ at fixed Gr, equivalent to the observed tendency toward the maximum of ϵ at given Re.

The apparent $Gr^{-1/2}$ scaling of the lower bound on γ in (69) and (70) at high Gr is reconciled with the observed turbulent Gr^0 scaling when the additional dependence on the Taylor microscale in (72) is accounted for. Recalling the defining relations for λ and β, $\lambda^2=\nu U^2/\epsilon$ and $\beta=\epsilon\ell/U^3$, it is easy to see that $\ell^2/\lambda^2=\gamma^{1/3}Gr^{1/2}\beta^{2/3}$. The high-$Gr$ limit of the lower bound on γ in (70) is then seen to be equivalent to the simple relationships

$$\gamma\geqslant C\beta\Longleftrightarrow Re\geqslant C^{1/3}Gr^{1/2} \tag{79}$$

(where now $C=(C_{-M}/D_M)^{3/2}$).

Turbulent flows are characterized by the appearance of a non-zero residual dissipation as $Re\to\infty$ (however we note that the appearance of such a residual dissipation as $\nu\to 0$ does not imply turbulence (Doering *et al.* 2000)). From (79) above we see that $Gr\to\infty$ ensures that $Re\to\infty$. Although we cannot ensure *a priori* that any particular applied body force shape function will produce turbulence as $Gr\to\infty$, if it does in the sense that β saturates to its $O(1)$ upper bound as the Reynolds number subsequently diverges, then the lower bounds on γ in (70) and (79) are precisely the observed Gr^0 scaling. Another way to express this is as follows: if the Taylor microscale achieves its 'turbulent' value at high Gr, i.e. if $\lambda\sim O(\nu^{1/2})$ as $\nu\to 0$, then the lower bound on γ is proportional to ν^0 implying the appearance of a residual dissipation in the vanishing viscosity limit.

To summarize, in this paper we have derived upper and lower bounds on the time-averaged energy dissipation rate in solutions of the Navier–Stokes equations with an applied body force in the absence of boundaries. The bounds require no assumptions on statistical correlations in the solutions, and are expressed in terms of either the magnitude of the resulting 'outer' velocity scale (or *Re*, or R_λ) or the magnitude of the driving body force (or *Gr*). Based on considerations of exact steady laminar solutions in the low-*Gr* and -*Re* and high-*Gr* limits, and experiments and direct numerical simulations of turbulent flows at high *Gr* and *Re*, we expect that these estimates cannot generally be improved – modulo prefactors and perhaps the aspect ratio dependence of the lower bounds in (66) and (67). Steady laminar Stokes-like flows tend to saturate the bounds on one side, while turbulence tends to saturate the scalings of the limits on the other side.

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Appendix. Basic inequalities

Poincare's inequality: Suppose $f(\boldsymbol{x})$ is a mean-zero, periodic function on the d-dimensional torus $[0, L]^d$. Then

$$\|f\| \leqslant \frac{L}{2\pi}\|\nabla f\|. \tag{A 1}$$

Proof. Write

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{f}_{\boldsymbol{k}}(t) \tag{A 2}$$

for $\boldsymbol{k} = (2\pi/L)\boldsymbol{n}$ where $\boldsymbol{n} = (n_1, \ldots, n_d)$ with integer n_i and $k = |\boldsymbol{k}| \geqslant 2\pi/L$, and consider the L^2 norm of f and ∇f in terms of their Fourier coefficients to see that

$$\|\nabla^N \boldsymbol{f}\|^2 = L^d \sum_{\boldsymbol{k}} |\hat{f}_{\boldsymbol{k}}|^2 \leqslant L^d \sum_{\boldsymbol{k}} \frac{L^2k^2}{4\pi^2}|\hat{f}_{\boldsymbol{k}}|^2 = \frac{L^2}{4\pi^2}\|\nabla \boldsymbol{f}\|^2. \tag{A 3}$$

□

Cauchy–Schwarz inequality: Suppose $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are square-integrable functions. Then

$$\left|\int fg\right|^2 \leqslant \int |f|^2 \int |g|^2. \tag{A 4}$$

Proof. (We establish it for real scalar functions; proofs for complex and/or vector-valued functions proceed similarly.) Assume $g \neq 0$, otherwise the assertion is obvious. Let t be a real parameter. Then since $(f - tg)^2 \geqslant 0$,

$$0 \leqslant \int (f - tg)^2 = \int f^2 - 2t \int fg + t^2 \int g^2. \tag{A 5}$$

To establish the result, minimize the right-hand side over t with the choice $t = \int fg / \int g^2$. □

Gronwall's inequality: Suppose there are real-valued functions $g(t)$ and $h(t)$ so that $f(t)$ and its derivative $\mathrm{d}f/\mathrm{d}t$ satisfy

$$\frac{\mathrm{d}f}{\mathrm{d}t} \leqslant g(t)f(t) + h(t). \tag{A 6}$$

Then for $t \geqslant 0$,

$$f(t) \leqslant f(0)\exp\left(\int_0^t g(s)\,\mathrm{d}s\right) + \int_0^t \exp\left(\int_r^t g(s)\,\mathrm{d}s\right) h(r)\,\mathrm{d}r. \tag{A 7}$$

Proof. Multiply (A 6) by the positive (to preserve the inequality) integrating factor $\exp(-\int_0^t g(s)\,\mathrm{d}s)$ and rewrite as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\exp\left(-\int_0^t g(s)\,\mathrm{d}s\right)\right) \leqslant h(t)\exp\left(-\int_0^t g(s)\,\mathrm{d}s\right). \tag{A 8}$$

Then integrate from 0 to t (preserving the inequality), multiply through by the positive factor $\exp(+\int_0^t g(s)\,\mathrm{d}s)$, and rearrange to establish the result. □

Hölder's inequality: We use just one simple version of Hölder's inequality, and for simplicity here prove it just for real scalar functions. Suppose $f(\boldsymbol{x})$ is integrable and $g(\boldsymbol{x})$ is bounded. Then

$$\left|\int fg\right| \leqslant \sup_{\boldsymbol{x}'} |g(\boldsymbol{x}')| \int |f|. \tag{A 9}$$

Proof. Pointwise,

$$\pm f(\boldsymbol{x})g(\boldsymbol{x}) \leqslant \sup_{\boldsymbol{x}'} |g(\boldsymbol{x}')||f(\boldsymbol{x})|. \tag{A 10}$$

Integration over $\boldsymbol{x}$ preserves the inequality. □

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